# MATH 3060 Assignment 5 solution

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1. Suppose $f$ is not uniformly continuous, then we can find a positive number $\epsilon$ and sequences $(x_n), (x'_n)$ in $E$ so that
$$d_X(x_n, x'_n) \to 0,$$
and
$$d_Y(f(x_n), f(x'_n)) \geq \epsilon.$$
Note that the two conditions above can pass to subsequences. Since $E$ is compact, we may assume $x_n \to x \in E$, but then
$$d_X(x'_n, x) \leq d_X(x'_n, x_n) + d_X(x_n, x) \to 0,$$
so we have $x'_n \to x$, and hence $f(x'_n) \to f(x)$. This is a contradiction because
$$\epsilon \leq d_Y(f(x_n), f(x'_n)) \leq d_Y(f(x_n), f(x)) + d_Y(f(x), f(x'_n)) \to 0.$$

2. Note that $f'_n(x) = nx^{n-1}$. For $x \in [0, \delta]$,
$$|f'_n(x)| \leq n\delta^{n-1} \to 0$$
as $n \to \infty$. In particular, $|f'_n|$ is uniformly bounded, so $f_n$ is equicontinuous on $[0, \delta]$.
On the other hand, taking $x_n = 2^{-1/n} < 1$. We have $x_n \to 1$ as $n \to 1$ but $|f_n(x_n) - f_n(1)| = \frac{1}{2}$, so $f_n$ is not equicontinous on $[0, 1]$.

3. We will show that the image is uniformly bounded and equicontinuous, then we can apply the Ascoli's theorem to conclude.
First of all, for $f \in C([0, 1])$ and $x \in [0, 1]$, we have
$$\begin{aligned} |Tf(x)| &= \left|\cos^2 x + \int_0^x \frac{f(t)}{1 + f^2(t)} dt\right| \\ &\leq 1 + \left|\int_0^x dt\right| \\ &= 1 + x \\ &\leq 2, \end{aligned}$$

and on the other hand, let $\epsilon > 0$. If we take $\delta_1 = \epsilon/4$, then for any $x, x' \in [0,1]$ with $|x - x'| < \delta_1$, we have $|\cos^2 x - \cos^2 x'| = 2|\cos\xi \sin\xi||x - x'| < \epsilon/2$ (Mean value theorem). And hence

$$\begin{aligned} |Tf(x) - Tf(x')| &< \frac{\epsilon}{2} + \left| \int_{x'}^{x} \frac{f(t)}{1 + f^2(t)} dt \right| \\ &\leq \frac{\epsilon}{2} + |x - x'| \\ &< \epsilon. \end{aligned}$$

So $T(C([0,1]))$ is equicontinuous.

4. Note that $K$ is bounded by some constant $M > 0$, $g$ is bounded by some constant $M' > 0$ and both of them are uniformly continuous by question 1.

   (a) Let $\epsilon > 0$, we choose $\delta > 0$ so that $|\lambda| M(b - a)\delta < \epsilon$. Then for $||f - f'|| < \delta$, we have

$$\begin{aligned} |T_\lambda f(x) - T_\lambda f'(x)| &= \left| \lambda \int_a^b K(x,t)(f(t) - f'(t)) dt \right| \\ &\leq \left| \lambda \int_a^b M\delta dt \right| \\ &< \epsilon. \end{aligned}$$

   (b) Let's assume $|f(x)| \leq L$ for any $f \in \mathcal{C}$. We need to show $T_\lambda(\mathcal{C})$ is bounded and equicontinuous. Boundedness follows from the definition of $T_\lambda$:

$$||T_\lambda f|| \leq |\lambda| ML(b - a) + M'.$$

   To show equicontinuity, let $\epsilon > 0$, we can choose $\delta_1 > 0$ so that $|g(x) - g(x')| < \epsilon/2$ whenever $|x - x'| < \delta_1$. Choose $\epsilon' > 0$ so that $|\lambda|(b - a)L\epsilon' < \epsilon/2$, we can also find $\delta_2 > 0$ so that $|K(x,t) - K(x',t')| < \epsilon$ whenever $||(x - x', t - t')|| < \delta_2$. Now, take $\delta = \min\{\delta_1, \delta_2\}$, if $|x - x'| < \delta$ and $f \in \mathcal{C}$, then

$$\begin{aligned} |T\lambda f(x) - T_\lambda f(x')| &< \left| \lambda \int_a^b (K(x,t) - K(x',t)) f(t) dt \right| + \frac{\epsilon}{2} \\ &\leq |\lambda|(b - a)L\epsilon' + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$